Formalizing 2-Adjoint Equivalences in Homotopy Type Theory

Based on joint work w/ J. Chang, C. Kapulkin, R. Sandford

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Contents

About this project

- NSERC - USRA project
- Collaborators
  - Jonathan Chang
  - Ryan Sandford
  - Supervisor: Chris Kapulkin
- Formalization found on Github
  - Contains formalizations of results in HoTT book, optimized proofs, and new material
  - gebner/hott3
- Paper forthcoming

Formalizing HoTT

- Formalization?
  - Coq, Agda
- We will be using Lean 3
  - HoTT for Lean 3 library
- Demo: ap and naturality of homotopies.

Lean and ap

- For $f : A \to B$ and $x, y : A$,

$$f[-] : (x = y) \to (fx = fy).$$

- For $p, q : x = y$,

$$f[\![-]\!] : (p = q) \to (f[p] = f[q]).$$

Lean and homotopies

- For $f, g : \prod_{x:A} Bx$,

$$f \sim g :\equiv \prod_{x:A} fx = gx.$$

- For $H : f \sim g$ and $x : A$,

$$H_x : fx = gx.$$

Proposition

For $f, g : A \to B$, $H : f \sim g$ and $p : x = y$, the following diagram commutes.

$$\begin{array}{ccc} fx & \xrightarrow{f[p]} & fy \\ {\scriptstyle H_x}\downarrow & = & \downarrow{\scriptstyle H_y} \\ gx & \xrightarrow[g[p]]{} & gy \end{array}$$

Using Univalence

Lemma (Equivalence Induction)

For $P : \prod_{A,B:\mathcal{U}}(A \simeq B) \to \mathcal{U}$ *and* $f : A \simeq B$,

$$P(A, A, \mathrm{id}_A) \to P(A, B, f).$$

Lemma (Based Homotopy Induction)

Given $f : A \to B$, *the types*

$$\sum_{g:A\to B} f \sim g \text{ and } \sum_{g:A\to B} g \sim f$$

are contractible with center $(f, \mathrm{refl_f})$.

Equivalence of types?

Definition

For $f : A \to B$, f has a *quasi-inverse* if:

$$\mathrm{qinv}\, f :\equiv \sum_{g:B\to A} (gf \sim \mathrm{id}_A) \times (fg \sim \mathrm{id}_B).$$

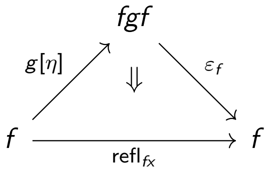
$$A \simeq B :\equiv \cancel{\sum_{f:A\to B} \mathrm{qinv}\, f}?$$ No good

Half Adjoint Equivalence

Definition

For $f : A \to B$, f is a *half adjoint equivalence* if:

$$\mathsf{ishadj}\ f :\equiv \sum_{g:B\to A} \ \sum_{\eta:gf\sim \mathsf{id}_A} \ \sum_{\varepsilon:fg\sim \mathsf{id}_B} f[\eta] \sim \varepsilon_f .$$

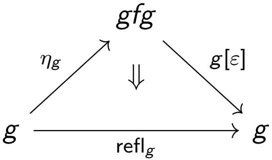


Left Half Adjoint Equivalence

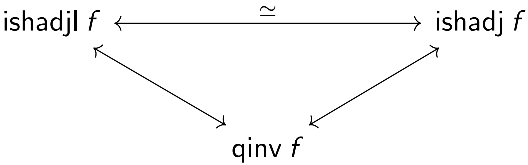
Definition

For $f : A \to B$, f is a *left half adjoint equivalence* if:

$$\text{ishadjl } f :\equiv \sum_{g:B\to A} \sum_{\eta:gf\sim \text{id}_A} \sum_{\varepsilon:fg\sim \text{id}_B} \eta_g = g[\varepsilon].$$



How they interact?



qinv is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$\text{qinv } f \simeq \prod_{x:A} x = x$$

Proof.

$$\begin{aligned}\text{qinv id}_A &\equiv \sum_{g:B\to A} (g \sim \text{id}_A) \times (g \sim \text{id}_A) \\ &\simeq \sum_{g:B\to A} \sum_{\eta:g\sim\text{id}_A} g \sim \text{id}_A \\ &\simeq \sum_{u:\sum_{g:B\to A} g\sim\text{id}_A} \text{pr}_1 u \sim \text{id}_A\end{aligned}$$

qinv is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$\mathsf{qinv}\ f \simeq \prod_{x:A} x = x$$

Proof.

$$\begin{aligned}\mathsf{qinv}\ \mathrm{id}_A &\simeq \sum_{u:\sum_{g:B\to A} g\sim \mathrm{id}_A} \mathrm{pr}_1 u \sim \mathrm{id}_A \\ &\simeq \mathrm{id}_A \sim \mathrm{id}_A \\ &\equiv \prod_{x:A} x = x. \quad \square\end{aligned}$$

qinv is not a proposition

Corollary

qinv id_{S^1} *is not a proposition.*

Proof.

By previous theorem, it suffices to show $\prod_{x:S^1} x = x$ is not a proposition. We know $\pi_1(S^1) = \mathbb{Z}$, so construct $h, h' : \prod_{x:S^1} x = x$ s.t.

$$h_{\mathrm{base}} = \mathrm{refl}_{\mathrm{base}}$$
$$h'_{\mathrm{base}} = \mathrm{loop} : \mathrm{base} = \mathrm{base}.$$

$\mathrm{refl}_{\mathrm{base}} \neq \mathrm{loop}$ so $h \neq h'$. □

From Quasi-inverses to Half Adjoint Equivalences

Theorem

For $f : A \to B$, ishadj f *is a proposition.*

Proof.

Assume ishadj f is inhabited.

$$
\begin{aligned}
\text{ishadj}\,\text{id}_A &\simeq \sum_{g:B\to A}\ \sum_{\eta:g\sim\text{id}_A}\ \sum_{\varepsilon:g\sim\text{id}_A} \text{id}_A[\eta] \sim \varepsilon \\
&\simeq \sum_{\varepsilon:\text{id}_A\sim\text{id}_A} \text{id}_A[\text{refl}] \sim \varepsilon \\
&\simeq \sum_{\varepsilon:\text{id}_A\sim\text{id}_A} \text{refl} \sim \varepsilon
\end{aligned}
$$

Apply based homotopy induction. □

Full Adjoint Equivalences

Definition

For $f : A \to B$, the data of a full-adjoint equivalence is

$$\text{adj}\, f :\equiv \sum_{g:B\to A} \; \sum_{\eta:gf\sim \text{id}_A} \; \sum_{\varepsilon:fg\sim \text{id}_B} f[\eta] \sim \varepsilon_f \times \eta_g \sim g[\varepsilon].$$

adj is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$\mathsf{adj}\ f \simeq \prod_{x:A} \mathsf{refl}_x = \mathsf{refl}_x .$$

Proof.

$$\begin{aligned}
\mathsf{adj}\ \mathsf{id}_A &\equiv \sum_{g:B\to A} \sum_{\eta:g\sim \mathsf{id}_A} \sum_{\varepsilon:g\sim \mathsf{id}_A} \mathsf{id}_A[\eta] \sim \varepsilon \times \eta_g \sim g[\varepsilon] \\
&\simeq \sum_{\varepsilon:\mathsf{id}_A\sim \mathsf{id}_A} \mathsf{id}_A[\mathsf{refl}] \sim \varepsilon \times \mathsf{refl} \sim \mathsf{id}_A[\varepsilon] \\
&\simeq \sum_{\varepsilon:\mathsf{id}_A\sim \mathsf{id}_A} \mathsf{refl} \sim \varepsilon \times \mathsf{refl} \sim \mathsf{id}_A[\varepsilon]
\end{aligned}$$

adj is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$\text{adj } f \simeq \prod_{x:A} \text{refl}_x = \text{refl}_x.$$

Proof.

$$\begin{aligned} \text{adj id}_A &\simeq \sum_{\varepsilon:\text{id}_A \sim \text{id}_A} \text{refl} \sim \varepsilon \times \text{refl} \sim \text{id}_A[\varepsilon] \\ &\simeq \sum_{\varepsilon:\text{id}_A \sim \text{id}_A} \sum_{\tau:\text{refl} \sim \varepsilon} \text{refl} \sim \text{id}_A[\varepsilon] \\ &\simeq \sum_{u:\sum_{\varepsilon:\text{id}_A \sim \text{id}_A} \text{refl} \sim \varepsilon} \text{refl} \sim \text{id}_A[\text{pr}_1 u] \end{aligned}$$

adj is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$\mathsf{adj}\ f \simeq \prod_{x:A} \mathsf{refl}_x = \mathsf{refl}_x.$$

Proof.

$$\begin{aligned} \mathsf{adj}\ \mathsf{id}_A &\simeq \sum_{u:\sum_{\varepsilon:\mathsf{id}_A \sim \mathsf{id}_A} \mathsf{refl} \sim \varepsilon} \mathsf{refl} \sim \mathsf{id}_A[\mathsf{pr}_1 u] \\ &\simeq \mathsf{refl} \sim \mathsf{id}_A[\mathsf{refl}] \\ &\equiv \mathsf{refl} \sim \mathsf{refl} \equiv \prod_{x:A} \mathsf{refl}_x = \mathsf{refl}_x. \quad \square \end{aligned}$$

adj is not a proposition

Corollary

adj id_{S^2} *is not a proposition.*

Proof.

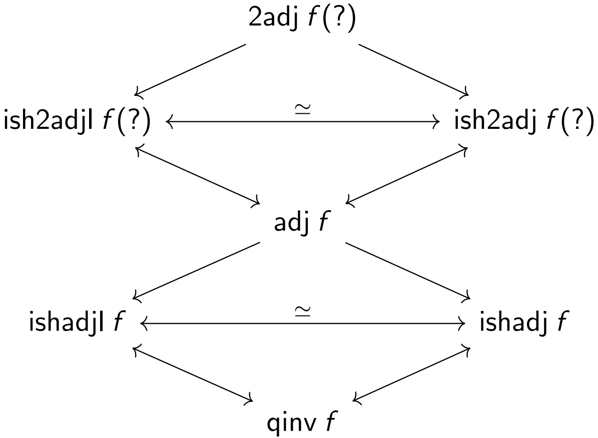
By previous theorem, it suffices to show $\prod_{x:S^2} \mathrm{refl}_x = \mathrm{refl}_x$ is not a proposition. We know $\pi_2(S^2) = \mathbb{Z}$, so construct $h, h' : \prod_{x:S^2} \mathrm{refl}_x = \mathrm{refl}_x$ s.t.

$$h_{\mathrm{base}} = \mathrm{refl}_{\mathrm{refl}_{\mathrm{base}}}$$
$$h'_{\mathrm{base}} = \mathrm{cell} : \mathrm{refl}_{\mathrm{base}} = \mathrm{refl}_{\mathrm{base}}.$$

$\mathrm{refl}_{\mathrm{refl}_{\mathrm{base}}} \neq \mathrm{cell}$ so $h \neq h'$. □

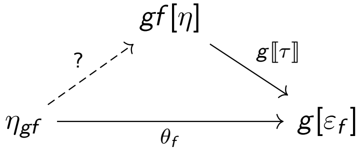
2-Adjoint Equivalences?

- What are 2-adjoint equivalences?



Finding the missing coherence

- Building blocks: $g[\![\tau]\!], \tau_g, f[\![\theta]\!], \theta_f$
- Candidate: ~~$g[\![\tau]\!] \sim \theta_f$?~~ Does not typecheck



Naturality coherence

Lemma

For $H : gf \sim \mathrm{id}_A$,

$$\mathrm{Coh}\, H : H_{gf} \sim gf[H].$$

Proof.

$$\begin{array}{ccc} gfgf & \xrightarrow{\;H_{gf}\;} & gf \\ {\scriptstyle gf[H]}\downarrow & = & \downarrow{\scriptstyle H} \\ gf & \xrightarrow[\;H\;]{} & \mathrm{id}_A \end{array}$$

□

Half 2-Adjoint Equivalences

Definition

For $f : A \to B$, f is a *half 2-adjoint equivalence* if

$$\text{ish2adj}\, f :\equiv \sum_{g:B\to A} \sum_{\eta:gf\sim\text{id}_A} \sum_{\varepsilon:fg\sim\text{id}_B} \sum_{\tau:f[\eta]\sim\varepsilon_f} \sum_{\theta:\eta_g\sim g[\varepsilon]} \text{Coh}\, \eta \cdot g[\![\tau]\!] \sim \theta_f$$

Left Half 2-Adjoint Equivalences

Definition

For $f : A \to B$, f is a *left half 2-adjoint equivalence* if

$$\text{ish2adj}\, f :\equiv \sum_{g:B\to A} \sum_{\eta:gf\sim\text{id}_A} \sum_{\varepsilon:fg\sim\text{id}_B} \sum_{\tau:f[\eta]\sim\varepsilon_f} \sum_{\theta:\eta_g\sim g[\varepsilon]} \tau_g \cdot \text{Coh}\,\varepsilon \sim f[\![\theta]\!]$$

Half 2-Adjoint Equivalences

Lemma

For $f : A \to B$ *with* $(g, \eta, \varepsilon, \theta) : \text{ishadjl } f$*,*

$$\sum_{\tau : f[\eta] \sim \varepsilon_f} \text{Coh } \eta \cdot g[\![\tau]\!] \sim \theta_{fx} \text{ is contractible.}$$

For $f : A \to B$ *with* $(g, \eta, \varepsilon, \tau) : \text{ishadj } f$*,*

$$\sum_{\theta : \eta_g \sim g[\varepsilon]} \tau_g \cdot \text{Coh } \varepsilon_y \sim f[\![\theta]\!] \text{ is contractible.}$$

Promoting to a Half 2-Adjoint Equivalence

Theorem

1. ishadjl $f \to$ ish2adj f
2. ishadj $f \to$ ish2adjl f

Proof.

Take missing coherences to be center of contraction. □

Corollary

1. adj $f \to$ ish2adj f
2. adj $f \to$ ish2adjl f

Proof.

Discard coherence and use above theorem. □

Half Two-Adjoint Equivalences are propositions

Theorem

For $f : A \to B$, ish2adj f *is a proposition.*

Proof.

Assume f is a 2-adjoint equivalence.

$$\begin{aligned}
\text{ish2adj } f &\equiv \sum_{g:B\to A} \sum_{\eta:gf\sim \mathrm{id}_A} \sum_{\varepsilon:fg\sim \mathrm{id}_B} \sum_{\tau:f[\eta]\sim\varepsilon_f} \sum_{\theta:\eta_g\sim g[\varepsilon]} \mathrm{Coh}\ \eta \cdot g[\![\tau]\!] \sim \theta_f \\
&\simeq \sum_{g:B\to A} \sum_{\eta:gf\sim \mathrm{id}_A} \sum_{\varepsilon:fg\sim \mathrm{id}_B} \sum_{\theta:\eta_g\sim g[\varepsilon]} \sum_{\tau:f[\eta]\sim\varepsilon_f} \mathrm{Coh}\ \eta \cdot g[\![\tau]\!] \sim \theta_f \\
&\simeq \sum_{h:\text{ishadjl } f} \sum_{\tau:f[\eta]\sim\varepsilon_f} \mathrm{Coh}\ \eta \cdot g[\![\tau]\!] \sim \theta_f
\end{aligned}$$

Half Two-Adjoint Equivalences are propositions

Theorem

For $f : A \to B$, $\mathsf{ish2adj}\ f$ is a proposition.

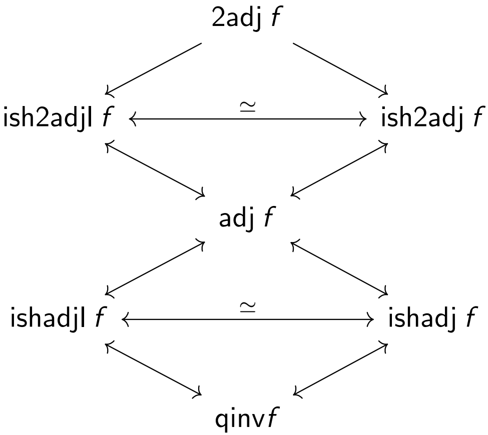
Proof.

$$\mathsf{ish2adj}\ f \simeq \sum_{h:\mathsf{ishadjl}\ f} \ \sum_{\tau : f[\eta] \sim \varepsilon_f} \mathsf{Coh}\ \eta \cdot g[\![\tau]\!] \sim \theta_f$$

$$\simeq \sum_{\tau : f[\eta] \sim \varepsilon_f} \mathsf{Coh}(h_\eta) \cdot (h_g)[\![\tau]\!] \sim (h_\theta)_f$$

Apply previous lemma. □

How they interact



Full 2-Adjoint Equivalence

Definition

For $f : A \to B$, the data of a *full 2-adjoint equivalence* is

$$\mathrm{2adj}\, f :\equiv \sum_{g:B\to A} \sum_{\eta: gf\sim \mathrm{id}_A} \sum_{\varepsilon: fg\sim \mathrm{id}_B} \sum_{\tau: f[\eta]\sim \varepsilon_f} \sum_{\theta:\eta_g \sim g[\varepsilon]} \mathrm{Coh}\,\eta \cdot g[\![\tau]\!] \sim \theta_f \times \tau_g \cdot \mathrm{Coh}\,\varepsilon \sim f[\![\theta]\!].$$

2adj is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$\mathsf{2adj}\ f \simeq \prod_{x:A} \mathsf{refl}_{\mathsf{refl}_x} = \mathsf{refl}_{\mathsf{refl}_x}.$$

Proof.

$$\mathsf{2adj}\ \mathsf{id}_A \equiv \sum_{g:B\to A} \sum_{\eta:g\sim\mathsf{id}_A} \sum_{\varepsilon:g\sim\mathsf{id}_A} \sum_{\tau:\mathsf{id}_A[\eta]\sim\varepsilon} \sum_{\theta:\eta_g\sim g[\varepsilon]}$$

$$\mathsf{Coh}\ \eta \cdot g[\![\tau]\!] \sim \theta \times \tau_g \cdot \mathsf{Coh}\ \varepsilon \sim \mathsf{id}_A[\![\theta]\!]$$

$$\simeq \sum_{\theta:\mathsf{refl}\sim\mathsf{refl}} \mathsf{refl}_{\mathsf{refl}} \sim \theta \times \mathsf{refl}_{\mathsf{refl}} \sim \mathsf{id}_A[\![\theta]\!]$$

2adj is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$2\text{adj } f \simeq \prod_{x:A} \text{refl}_{\text{refl}_x} = \text{refl}_{\text{refl}_x}.$$

Proof.

$$\begin{aligned} 2\text{adj id}_A &\simeq \sum_{\theta:\text{refl}\sim\text{refl}} \text{refl}_{\text{refl}} \sim \theta \times \text{refl}_{\text{refl}} \sim \text{id}_A[\![\theta]\!] \\ &\simeq \sum_{\theta:\text{refl}\sim\text{refl}} \sum_{\mathcal{A}:\text{refl}_{\text{refl}}\sim\theta} \text{refl}_{\text{refl}} \sim \theta \\ &\simeq \sum_{u:\sum_{\theta:\text{refl}\sim\text{refl}} \text{refl}_{\text{refl}}\sim\theta} \text{refl}_{\text{refl}} \sim \text{pr}_1 u \end{aligned}$$

2adj is not a proposition

Theorem

For $f : A \to B$, an equivalence,

$$\mathrm{2adj}\, f \simeq \prod_{x:A} \mathrm{refl}_{\mathrm{refl}_x} = \mathrm{refl}_{\mathrm{refl}_x}.$$

Proof.

$$\begin{aligned}
\mathrm{2adj}\, \mathrm{id}_A &\simeq \sum_{u:\sum_{\theta:\mathrm{refl}\sim\mathrm{refl}} \mathrm{refl}_{\mathrm{refl}}\sim\theta} \mathrm{refl}_{\mathrm{refl}} \sim \mathrm{pr}_1 u \\
&\simeq \mathrm{refl}_{\mathrm{refl}} \sim \mathrm{refl}_{\mathrm{refl}} \\
&\equiv \prod_{x:A} \mathrm{refl}_{\mathrm{refl}_x} = \mathrm{refl}_{\mathrm{refl}_x}. \qquad \square
\end{aligned}$$

2adj is not a proposition

Corollary

2adj id_{S^3} *is not a proposition.*

Proof.

By previous theorem, it suffices to show $\prod_{x:S^3} \mathrm{refl}_{\mathrm{refl}_x} = \mathrm{refl}_{\mathrm{refl}_x}$ is not a proposition. We know $\pi_3(S^3) = \mathbb{Z}$, so construct $h, h' : \prod_{x:S^3} \mathrm{refl}_{\mathrm{refl}_x} = \mathrm{refl}_{\mathrm{refl}_x}$ s.t.

$$h_{\mathrm{base}} = \mathrm{refl}_{\mathrm{refl}_{\mathrm{refl}_{\mathrm{base}}}}$$
$$h'_{\mathrm{base}} = \mathit{cell} : \mathrm{refl}_{\mathrm{refl}_{\mathrm{base}}} = \mathrm{refl}_{\mathrm{refl}_{\mathrm{base}}}.$$

$\mathrm{refl}_{\mathrm{refl}_{\mathrm{refl}_{\mathrm{base}}}} \neq$ cell so $h \neq h'$. □

Summary

- Modularity starts at qinv
- Formalization found on Github : 528 lines
  - gebner/hott3
  - Paper forthcoming
- Sementical argument

Thank you!